Critical slowing down of the kinetic Gaussian model on hierarchical lattices

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The critical slowing down of the kinetic Gaussian model on hierarchical lattices is studied by means of a real-space time-dependent renormalization-group transformation. The dynamic critical exponent z and the exponent Δ are calculated. For hierarchical lattices with reducible generators, both the dynamic critical exponent z and the exponent Δ are independent of the fractal dimension D_f of the lattice, the number of branches m, and the number of bonds per branch b of the generator—i.e., $z=2$ and $\Delta=1$. For hierarchical lattices with irreducible generators, the exponent Δ is the same—i.e., $\Delta=1$; however, the dynamic critical exponent z is dependent on the concrete geometrical structure of these lattices. In addition, it was found that the lattice dependence of the correlation-length critical exponent ν is the same as that of the dynamic critical exponent z. Finally we give a brief discussion about universality for critical dynamics.

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I. INTRODUCTION

In dynamic critical phenomena, the critical slowing down is characterized by a divergent relaxation time τ of systems near the critical point. This divergence can be described by the dynamic critical exponent z which is given by

$$\tau \sim \xi^z \sim |T - T_c|^{-\Delta}, \tag{1}$$

where the exponent $\Delta = z\nu$, ν is the correlation-length critical exponent, T_c is the critical temperature, and ξ is the correlation length which also diverges at the critical point.

As we know, Glauber [1] introduced the single-spin flipping mechanism and exactly solved the one-dimensional kinetic Ising model. Also, Zhu and Yang [2] exactly solved the kinetic Gaussian model on translation-invariant lattices by presenting a single-spin transition mechanism, which is a generalization of Glauber's single-spin flipping mechanism and is suitable to describe both discrete and continuous spin systems. In addition, other efforts to obtain the dynamic critical exponent z have been made by means of a variety of approaches, including ε expansion [3,4], the real-space time-dependent renormalization-group (TDRG) transformation [5–8], high-temperature series expansion [9,10], Monte Carlo simulations [11–14], damage spreading [15,16], and Monte Carlo renormalization-group calculations [17,18]. As far as the TDRG transformation is concerned, it is a very effective method to study the critical slowing down of spin systems on fractal lattices [19–25] due to their self-similar geometrical features. Among these fractals, hierarchical lattices [26–29] are highly inhomogeneous, and they may provide insights into other low-symmetry problems such as random magnets, surfaces, etc. Therefore, a lot of work on hierarchical lattices has been done recently [30–34]. However, in comparison with the static critical behavior, much less attention has been paid to the study of the critical dynamics on these lattices.

Hierarchical lattices can be constructed through an iterative decoration of a two-point bond by a generator, which has two vertices—i.e., the nodes. Figure 1 shows the constructions of a few members of hierarchical lattices. First, one replaces a single bond by a basic cell, which is called a generator [see Figs. 1(b), 1(e), and 1(h)], and gets a lattice of order 1. At the next stage, each bond of the lattice of order 1 is decorated with the generator in the same manner, then the lattice of order 2 [see Figs. 1(c), 1(f), and 1(i)] is formed. Repetition of this procedure N times produces the lattice of order N. Finally, an infinite lattice—i.e., a hierarchical

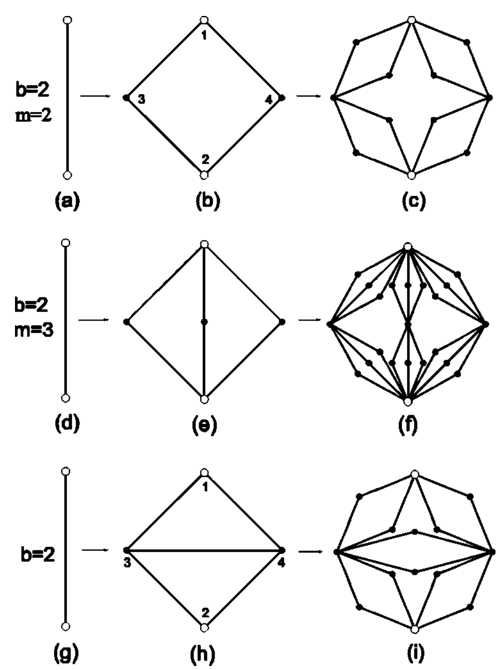

FIG. 1. First two stages of the constructions of a few members of hierarchical lattices.

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lattice—can be formed. According to graph theory [35,36], one can distinguish between a reducible generator and an irreducible one. A generator is reducible if it can be constructed from its constituent generators; otherwise, it is irreducible. Usually, a reducible generator is made up of some identical one-dimensional generators, and its two nodes are joined by m branches of b bonds. For example, the two generators shown in Figs. 1(b) and 1(e) are reducible because they can be constructed from two and three identical one-dimensional generators, respectively. In contrast with the above two reducible generators, the generator shown in Fig. 1(h) is irreducible because its two internal sites 3 and 4 are joined by a bond. In terms of recursion relations of the renormalization-group transformation, the recursion relation for a reducible generator can always be written as a product of the recursion relations for its one-dimensional generators. For example, consider the Ising model on a hierarchical lattice with the reducible generator shown in Fig. 1(b). One can write the recursion relation of the renormalization-group transformation as

$$Ae^{K'\sigma_1\sigma_2} = \left(\sum_{\sigma_3} e^{K\sigma_3(\sigma_1+\sigma_2)}\right)\left(\sum_{\sigma_4} e^{K\sigma_4(\sigma_1+\sigma_2)}\right),$$

where K denotes the interaction between the nearest-neighbor spins, K' represents the renormalized interaction, and A is an additive constant produced after the renormalization-group transformation. To describe the geometrical features of the generator of a hierarchical lattice, one can employ some parameters such as the aggregation number A, the minimum cut C, and the distance between the nodes, b, the definitions of which are as follows: the aggregation number A is the total number of bonds, the minimum cut C is the minimum number of bonds which need be cut to separate the nodes, and the distance between the nodes, b, is defined as the number of bonds on the shortest path joining them. Based on the above three parameters, the fractal dimension D_f and the connectivity Q are defined, respectively, as

$$D_f = \frac{\ln A}{\ln b}, \quad Q = \frac{\ln C}{\ln b}. \tag{2}$$

Kong and his collaborators [37] investigated phase transitions of the Gaussian model on a family of hierarchical lattices with reducible generators and calculated six static critical exponents, the results of which are as follows: $\alpha=(4-D_f)/2$, $\beta=(D_f-2)/4$, $\gamma=1$, $\delta=(D_f+2)/(D_f-2)$, $\eta=0$, and $\nu=1/2$. It can be found that the critical exponents γ, η, and ν are independent of the fractal dimension D_f of the lattice, while the others—i.e., α, β, and δ—are dependent on it.

In this paper, we study the critical slowing down of the kinetic Gaussian model on hierarchical lattices by means of a real-space time-dependent renormalization-group transformation. The dynamic critical exponent z and the exponent Δ are calculated. For hierarchical lattices with reducible generators, both the dynamic critical exponent z and the exponent Δ are independent of the fractal dimension D_f, the number of branches m, and the number of bonds per branch, b, of the generator—i.e., $z=2$ and $\Delta=1$. For hierarchical lattices with irreducible generators, the exponent Δ is the same—i.e., $\Delta=1$; however, the dynamic critical exponent z is dependent on the concrete geometrical structure of these lattices and can be different even if the fractal dimension D_f and the connectivity Q are the same. In the following section, we present descriptions of the kinetic Gaussian model with the single-spin transition mechanism and the method of the TDRG transformation. The critical slowing down of the kinetic Gaussian model on hierarchical lattices with reducible generators is studied in Sec. III. Section IV is devoted to hierarchical lattices with irreducible generators. Finally, we give a brief discussion and conclusion in Sec. V.

II. KINETIC GAUSSIAN MODEL AND TDRG TRANSFORMATION

Berlin and Kac [38] introduced the spherical and Gaussian models to illuminate some of the complexities of the critical phenomena, and they also pointed out the connection between the above two models by means of a calculation of the average of the fourth power of spin σ_j—i.e., $\langle\sigma_j^4$. In the spherical model, the spherical condition $\Sigma_{j=1}^{j=N}\sigma_j^2=N$ must be satisfied for every configuration of spins. Above the critical temperature, the distribution of a finite number of spins—i.e., a number independent of N for large N—is Gaussian. Below the critical temperature, however, deviations from a Gaussian distribution are obtained because of the cooperation among the spins, which is forced by the spherical condition and does not exist in the Gaussian model. In addition, the spherical model is a valid model for all temperatures, whereas the Gaussian model becomes invalid for temperatures below a certain critical temperature. Herein we shall restrict ourselves to the Gaussian model.

The reduced Hamiltonian of the Gaussian model can be expressed as

$$\mathcal{H} = -\beta H = K\sum_{\langle ij\rangle} \sigma_i\sigma_j, \tag{3}$$

where $\beta=1/k_BT$, K is the reduced interaction between the nearest-neighbor spins, the continuous spin variable σ takes any real value in the interval $(-\infty,+\infty)$, and the sum $\Sigma_{\langle ij\rangle}$ is over all nearest-neighbor pairs of spins. The probability of finding a given spin σ_i between σ_i and $\sigma_i+d\sigma_i$ is given by

$$f(\sigma_i)d\sigma_i \sim \exp\left(-\frac{b_{q_i}}{2}\sigma_i^2\right)d\sigma_i, \tag{4}$$

in which q_i is the coordination number of lattice site i and b_{q_i} is a constant only dependent on the coordination number q_i. Thus, the equilibrium probability distribution of the spins can be written as

$$P_e(\{\sigma\})\prod_i d\sigma_i = \frac{1}{Z}\exp[\mathcal{H}(\{\sigma\})]\prod_i d\sigma_i, \tag{5}$$

where $\{\sigma\}$ denotes a configuration of spins $(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_N)$, and Z is the partition function of system–i.e.,

$$Z = \int_{-\infty}^{\infty} \left[\prod_i d\sigma_i f(\sigma_i) \right] \exp[\mathcal{H}(\{\sigma\})]. \tag{6}$$

Suppose that the system, under a constraint, is first brought into an equilibrium state. Then at time $t=0$ the constraint is removed, and the system relaxes towards equilibrium via an interaction with a heat bath. In this process, only one spin is allowed to change itself each time according to the single-spin transition mechanism [2]. $P_e(\{\sigma\})$ can be regarded as the infinite-time limit of the spin time-dependent probability distribution $P(\{\sigma\},t)$, the time evolution of which is given by the master equation

$$\tau_0 \frac{d}{dt} P(\{\sigma\},t) = \sum_i \sum_{\hat{\sigma}_i} [- W_i(\sigma_i \to \hat{\sigma}_i) P(\{\sigma\},t) + W_i(\hat{\sigma}_i \to \sigma_i) P(\{\sigma_{j\neq i}\},\hat{\sigma}_i,t)], \tag{7}$$

where τ_0 is a bare time scale characterizing the coupling to a heat bath and $W_i(\sigma_i \to \hat{\sigma}_i)$ is the single-spin transition probability rate and satisfies the detailed balance condition

$$\frac{W_i(\sigma_i \to \hat{\sigma}_i)}{W_i(\hat{\sigma}_i \to \sigma_i)} = \frac{P_e(\sigma_1, \ldots, \hat{\sigma}_i, \ldots, \sigma_N)}{P_e(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N)}, \tag{8}$$

as well as the normalization condition

$$\sum_{\hat{\sigma}_i} W_i(\sigma_i \to \hat{\sigma}_i) = 1. \tag{9}$$

Usually, the spin transition probability rate takes the following form:

$$W_i(\sigma_i \to \hat{\sigma}_i) = \frac{1}{Q_i} \exp\left[K\hat{\sigma}_i \sum_{j(i)} \sigma_{j(i)} \right], \tag{10}$$

where $j(i)$ represents the set of all nearest neighbors to site i and the normalization factor Q_i can be expressed as

$$Q_i = \int_{-\infty}^{+\infty} \exp\left[K\hat{\sigma}_i \sum_{j(i)} \sigma_{j(i)} \right] f(\hat{\sigma}_i) d\hat{\sigma}_i = \exp\left[\frac{K^2}{2b_{q_i}} \left(\sum_{j(i)} \sigma_{j(i)} \right)^2 \right]. \tag{11}$$

To study the critical slowing down of the kinetic Gaussian model on hierarchical lattices, we shall use the real-space time-dependent renormalization-group transformation, proposed by Achiam and Kosterlitz [5] and restrict ourselves to the relaxation of an infinitely small perturbation from equilibrium. If we adopt a magneticlike perturbation in the following form at any time

$$P(\{\sigma\},t) = \left(1 + \sum_i h_{q_i}(t) \sigma_i \right) P_e(\{\sigma\}), \tag{12}$$

where the perturbation field $h_{q_i}(t)$ is dependent on the coordination number q_i of lattice site i, then master equation (7) becomes

$$\tau_0 \frac{d}{dt} \sum_i h_{q_i}(t) \sigma_i P_e(\{\sigma\}) = - \sum_i \sum_{\hat{\sigma}_i} h_{q_i}(t) (\sigma_i - \hat{\sigma}_i) W_i(\sigma_i \to \hat{\sigma}_i) P_e(\{\sigma\}). \tag{13}$$

By calculating that

$$\sum_{\hat{\sigma}_i} (\sigma_i - \hat{\sigma}_i) W_i(\sigma_i \to \hat{\sigma}_i) = \int_{-\infty}^{+\infty} (\sigma_i - \hat{\sigma}_i) W_i(\sigma_i \to \hat{\sigma}_i) f(\hat{\sigma}_i) d\hat{\sigma}_i = \sigma_i - \frac{K}{b_{q_i}} \sum_{j(i)} \sigma_{j(i)}, \tag{14}$$

master equation (13) can be reduced to the following form:

$$\tau_0 \frac{d}{dt} \sum_i h_{q_i}(t) \sigma_i P_e(\{\sigma\}) = - \sum_i h_{q_i}(t) \left(\sigma_i - \frac{K}{b_{q_i}} \sum_{j(i)} \sigma_{j(i)} \right) P_e(\{\sigma\}). \tag{15}$$

The TDRG transformation is composed of two stages. One is the rescaling of the space—i.e.,

$$x \to x' = bx, \tag{16}$$

where b is the length-rescaling factor. The other is the rescaling of the time scale—i.e.,

$$\tau_0' = b^z \tau_0. \tag{17}$$

Owing to the geometrical features of hierarchical lattices, the first stage of the TDRG transformation is carried out by means of the decimation renormalization-group transformation $\mathcal{R}$, which takes the form $\mathcal{R} \equiv \Sigma_{\{\sigma\}} T(\mu,\sigma) = \Sigma_{\{\sigma\}} \Pi_j \delta(\mu_j - \sigma_j)$, where σ_j denotes those spins retained in the process of the renormalization-group transformation. The renormalization-group transformation of master equation (15) gives

$$\tau_0 \frac{d}{dt} \mathcal{R} \left[\sum_i h_{q_i}(t) \sigma_i P_e(\{\sigma\}) \right] = - \mathcal{R} \left[\sum_i h_{q_i}(t) \left(\sigma_i - \frac{K}{b_{q_i}} \sum_{j(i)} \sigma_{j(i)} \right) P_e(\{\sigma\}) \right]. \tag{18}$$

In the invariant subspace of the parameter space $(K,\mathbf{h})$, the left-hand side of Eq. (18) gives rise to the following recursion relations:

$$K' = R(K), \quad \mathbf{h}'(t) = \mathbf{\Lambda} \mathbf{h}(t), \tag{19}$$

in which

$$\mathbf{h}(t) = \begin{pmatrix} h_{q_1}(t) \\ h_{q_2}(t) \\ \vdots \end{pmatrix}, \quad \mathbf{h}'(t) = \begin{pmatrix} h'_{q_1}(t) \\ h'_{q_2}(t) \\ \vdots \end{pmatrix},$$

and $\mathbf{\Lambda}$ is a transformation matrix. Also, the right-hand side of Eq. (18) results in the recursion relation

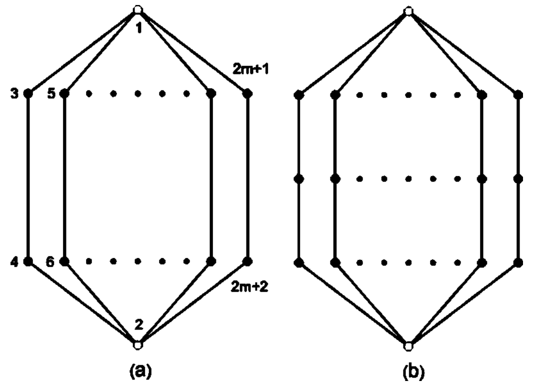

FIG. 2. Two families of reducible generators of hierarchical lattices: (a) the number of bonds per branch $b=3$ and (b) the number of bonds per branch $b=4$.

$$\mathbf{h}''(t) = \mathbf{\Omega}\mathbf{h}(t), \tag{20}$$

where

$$\mathbf{h}''(t) = \begin{pmatrix} h''_{q_1}(t) \\ h''_{q_2}(t) \\ \vdots \end{pmatrix}$$

and $\mathbf{\Omega}$ is a transformation matrix. In the second stage of the TDRG transformation, representing $\mathbf{h}''(t)$ in terms of $\mathbf{h}'(t)$ and performing the rescaling of the time scale will restore the master equation to an invariant form. As far as the dynamic critical exponent z is concerned, it can be determined in the following way [8]: if matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ commute, then

$$\frac{\lambda_{\max}}{\omega_{\max}} = b^z, \tag{21}$$

where $\lambda_{\max}$ and $\omega_{\max}$ are the largest eigenvalues of $\mathbf{\Lambda}$ and $\mathbf{\Omega}$, respectively; otherwise,

$$\frac{\lambda_{\max}}{\omega_{\min}} = b^z, \tag{22}$$

where $\omega_{\min}$ is the smallest eigenvalue of $\mathbf{\Omega}$. In addition, because a hierarchical lattice is highly inhomogeneous and its maximum order of ramification is infinite, it is necessary to present the following assumption [25,37]:

$$\frac{h_{q_i}(t)}{h_{q_j}(t)} = \frac{b_{q_j}}{b_{q_i}} = \frac{q_i}{q_j}, \tag{23}$$

which makes the TDRG transformation tractable in this case.

III. KINETIC GAUSSIAN MODEL ON HIERARCHICAL LATTICES WITH REDUCIBLE GENERATORS

As defined in Sec. I, a reducible generator consists of m branches of b bonds, which meet at the two nodes. Figure 2 shows two families of reducible generators which correspond to the cases of $b=3$ and $b=4$. For the hierarchical lattice with a reducible generator in the case of $b=3$ as well as an arbitrary number of branches m [see Fig. 2(a)], the fractal dimension D_f is equal to $1+\ln m/\ln 3$. Using assumption (23), master equation (15) becomes

$$\begin{aligned}\tau_0 \frac{d}{dt}\sum_\alpha \Bigg[& h_{mn_1}(t)\frac{\sigma_1^\alpha}{n_1} + h_{mn_2}(t)\frac{\sigma_2^\alpha}{n_2} \\ & + h_2(t)\sum_{r=1}^{m}(\sigma_{2r+1}^\alpha + \sigma_{2r+2}^\alpha)\Bigg]P_e(\{\sigma\}) \\ = -\sum_\alpha\left(1-\frac{2K}{b_2}\right)\Bigg[& h_{mn_1}(t)\frac{\sigma_1^\alpha}{n_1} + h_{mn_2}(t)\frac{\sigma_2^\alpha}{n_2} \\ & + h_2(t)\sum_{r=1}^{m}(\sigma_{2r+1}^\alpha + \sigma_{2r+2}^\alpha)\Bigg]P_e(\{\sigma\}),\end{aligned} \tag{24}$$

where α denotes the αth generator, the sum Σ_α is over all generators of the hierarchical lattice, the coefficient $1/n_1$ (or $1/n_2$) in the term σ_1^α (or σ_2^α) comes from the fact that n_1 (or n_2) neighboring generators share the same lattice site 1 (or 2), and the equilibrium probability distribution $P_e(\{\sigma\})$ can be expressed as

$$\begin{aligned}P_e(\{\sigma\}) &= \frac{1}{Z}\exp\left(K\sum_{\langle ij\rangle}\sigma_i\sigma_j - \sum_i \frac{b_{q_i}}{2}\sigma_i^2\right) \\ &= \frac{1}{Z}\prod_\alpha \exp\Bigg\{K\sum_{r=1}^{m}(\sigma_1^\alpha\sigma_{2r+1}^\alpha + \sigma_{2r+1}^\alpha\sigma_{2r+2}^\alpha + \sigma_{2r+2}^\alpha\sigma_2^\alpha) \\ &\quad - \frac{b_{mn_1}}{2}\frac{(\sigma_1^\alpha)^2}{n_1} - \frac{b_{mn_2}}{2}\frac{(\sigma_2^\alpha)^2}{n_2} \\ &\quad - \frac{b_2}{2}\sum_{r=1}^{m}\left[(\sigma_{2r+1}^\alpha)^2 + (\sigma_{2r+2}^\alpha)^2\right]\Bigg\}.\end{aligned} \tag{25}$$

Master equation (24) can also be written in the following form:

$$\tau_0\frac{d}{dt}\sum_i h_{q_i}(t)\sigma_i P_e(\{\sigma\}) = -\sum_i\left(1-\frac{2K}{b_2}\right)h_{q_i}(t)\sigma_i P_e(\{\sigma\}). \tag{26}$$

First, we perform the decimation renormalization-group transformation $\mathcal{R}$ to the equilibrium probability distribution $P_e(\{\sigma\})$—i.e., expression (25). Using assumption (23), one can get

$$\begin{aligned}\mathcal{R}\{P_e(\{\sigma\})\} = \frac{1}{Z}\prod_\alpha \Bigg\{ & \exp\left(-\frac{b_{mn_1}}{2}\frac{(\mu_1^\alpha)^2}{n_1} - \frac{b_{mn_2}}{2}\frac{(\mu_2^\alpha)^2}{n_2}\right) \\ & \times\prod_{r=1}^{m}\int_{-\infty}^{+\infty} d\sigma_{2r+1}^\alpha d\sigma_{2r+2}^\alpha \\ & \times\exp\Bigg(K(\mu_1^\alpha\sigma_{2r+1}^\alpha + \sigma_{2r+1}^\alpha\sigma_{2r+2}^\alpha + \sigma_{2r+2}^\alpha\mu_2^\alpha) \\ & -\frac{b_2}{2}[(\sigma_{2r+1}^\alpha)^2 + (\sigma_{2r+2}^\alpha)^2]\Bigg)\Bigg\}\end{aligned}$$

$$= \frac{1}{Z}\prod_{\alpha} A_0 \exp\left[K' \mu_1'^{\alpha}\mu_2'^{\alpha} - \frac{b_{n_1}}{2}\frac{1}{n_1}(\mu_1'^{\alpha})^2 - \frac{b_{n_2}}{2}\frac{1}{n_2}(\mu_2'^{\alpha})^2\right]$$

$$= \frac{1}{Z'} \exp\left(K' \sum_{\langle ij\rangle} \mu_i'\mu_j' - \sum_i \frac{b_{q_i}}{2}\mu_i'^2\right)$$

$$= P_e'(\{\mu'\}), \tag{27}$$

where

$$A_0 = \left(\frac{2\pi}{\sqrt{b_2^2 - K^2}}\right)^m, \tag{28}$$

the spins are rescaled as

$$\mu' = \xi(K)\mu = \sqrt{\frac{m(b_2^2 - 3K^2)}{b_2^2 - K^2}}\mu, \tag{29}$$

and the recursion relation of the renormalization-group transformation is given by

$$K' = R(K) = \frac{K^3}{b_2^2 - 3K^2}, \tag{30}$$

from which one can get the critical point $K_c = b_2/2$. Second, we perform the decimation renormalization-group transformation $\mathcal{R}$ to $\sigma_i P_e(\{\sigma\})$. Using assumption (23), one can get

$$\mathcal{R}\{\sigma_1^{\alpha} P_e(\{\sigma\})\} = \frac{1}{\xi(K)}\mu_1'^{\alpha} P_e'(\{\mu'\}), \tag{31}$$

$$\mathcal{R}\{\sigma_2^{\alpha} P_e(\{\sigma\})\} = \frac{1}{\xi(K)}\mu_2'^{\alpha} P_e'(\{\mu'\}), \tag{32}$$

$$\mathcal{R}\{\sigma_{2r+1}^{\alpha} P_e(\{\sigma\})\} = \frac{K(b_2\mu_1'^{\alpha} + K\mu_2'^{\alpha})}{\xi(K)(b_2^2 - K^2)} P_e'(\{\mu'\}) \quad (r = 1,2,\ldots,m), \tag{33}$$

and

$$\mathcal{R}\{\sigma_{2r+2}^{\alpha} P_e(\{\sigma\})\} = \frac{K(K\mu_1'^{\alpha} + b_2\mu_2'^{\alpha})}{\xi(K)(b_2^2 - K^2)} P_e'(\{\mu'\}) \quad (r = 1,2,\ldots,m). \tag{34}$$

Thus, using assumption (23) and Eqs. (31)–(34), the renormalization-group transformation of the left-hand side of master equation (26) gives

$$\tau_0 \frac{d}{dt}\mathcal{R}\left\{\sum_i h_{q_i}(t)\sigma_i P_e(\{\sigma\})\right\}$$

$$= \tau_0 \frac{d}{dt}\mathcal{R}\left\{\sum_{\alpha}\left[h_{mn_1}(t)\frac{\sigma_1^{\alpha}}{n_1} + h_{mn_2}(t)\frac{\sigma_2^{\alpha}}{n_2} + h_2(t)\sum_{r=1}^{m}(\sigma_{2r+1}^{\alpha} + \sigma_{2r+2}^{\alpha})\right] P_e(\{\sigma\})\right\}$$

$$= \tau_0 \frac{d}{dt}\sum_{\alpha}\frac{m(b_2+K)}{\xi(K)(b_2-K)}\left(h_{n_1}(t)\frac{\mu_1'^{\alpha}}{n_1} + h_{n_2}(t)\frac{\mu_2'^{\alpha}}{n_2}\right) P_e'(\{\mu'\})$$

$$= \tau_0 \frac{d}{dt}\sum_i \frac{m(b_2+K)}{\xi(K)(b_2-K)} h_{q_i}(t)\mu_i' P_e'(\{\mu'\})$$

$$= \tau_0 \frac{d}{dt}\sum_i h_{q_i}'(t)\mu_i' P_e'(\{\mu'\}), \tag{35}$$

where

$$h_{q_i}'(t) = \lambda h_{q_i}(t) = \frac{m(b_2+K)}{\xi(K)(b_2-K)} h_{q_i}(t). \tag{36}$$

In the same way, the renormalization-group transformation of the right-hand side of master equation (26) gives

$$-\mathcal{R}\left\{\sum_i \left(1 - \frac{2K}{b_2}\right) h_{q_i}(t)\sigma_i P_e(\{\sigma\})\right\}$$

$$= -\mathcal{R}\left\{\sum_{\alpha}\left(1 - \frac{2K}{b_2}\right)\left[h_{mn_1}(t)\frac{\sigma_1^{\alpha}}{n_1} + h_{mn_2}(t)\frac{\sigma_2^{\alpha}}{n_2} + h_2(t)\sum_{r=1}^{m}(\sigma_{2r+1}^{\alpha} + \sigma_{2r+2}^{\alpha})\right] P_e(\{\sigma\})\right\}$$

$$= -\sum_i \frac{m(b_2 - 2K)(b_2+K)}{\xi(K) b_2 (b_2 - K)} h_{q_i}(t)\mu_i' P_e'(\{\mu'\})$$

$$= -\sum_i \left(1 - \frac{2K'}{b_2}\right) h_{q_i}''(t)\mu_i' P_e'(\{\mu'\}), \tag{37}$$

where

$$h_{q_i}''(t) = \omega h_{q_i}(t) = \frac{m(b_2 - 2K)(b_2 + K)}{\xi(K)(b_2 - K)(b_2 - 2K')} h_{q_i}(t). \tag{38}$$

Furthermore, if we represent $h_{q_i}''(t)$ in terms of $h_{q_i}'(t)$ and perform the rescaling of the time scale by

$$\tau_0' = b^z \tau_0 = \frac{\lambda}{\omega}\tau_0 = \frac{(b_2 + K)^2}{b_2^2 - 3K^2}\tau_0, \tag{39}$$

then the invariant form of master equation (26) can be restored—i.e.,

$$\tau_0' \frac{d}{dt}\sum_i h_{q_i}'(t)\mu_i' P_e'(\{\mu'\}) = -\sum_i \left(1 - \frac{2K'}{b_2}\right) h_{q_i}'(t)\mu_i' P_e'(\{\mu'\}). \tag{40}$$

From the recursion relation of the renormalization-group transformation—i.e., Eq. (30)—one can obtain the correlation-length critical exponent ν—i.e.,

$$\frac{1}{\nu} = \frac{1}{\ln b} \ln \left.\left(\frac{dK'}{dK}\right)\right|_{K_c} = \frac{\ln 9}{\ln 3} = 2. \tag{41}$$

Also, from Eq. (39) the dynamic critical exponent z can be calculated as

$$z = \frac{1}{\ln b} \ln \left. \frac{(b_2 + K)^2}{b_2^2 - 3K^2} \right|_{K_c} = \frac{\ln 9}{\ln 3} = 2. \tag{42}$$

Thus, the above two equations (41) and (42) give

$$\Delta = z\nu = 1. \tag{43}$$

For the hierarchical lattice with a reducible generator in the case of $b=4$ as well as an arbitrary number of branches m [see Fig. 2(b)], the fractal dimension D_f is equal to $1 + \ln m / \ln 4$. By means of the TDRG transformation, one can also obtain that

$$z = 2, \quad \Delta = 1. \tag{44}$$

The above calculations, as well as the investigation on a family of diamond-type hierarchical lattices with the number of bonds per branch $b=2$ [24], reveal that for the kinetic Gaussian model on hierarchical lattices with reducible generators, both the dynamic critical exponent z and the exponent Δ are independent of the fractal dimension D_f, the number of branches m, and the number of bonds per branch b of the generator—i.e., $z=2$ and $\Delta=1$. This result is very similar to that in Ref. [2], where it has been found that for one-, two-, and three-dimensional translation-invariant lattices, both the dynamic critical exponent $z=2$ and the exponent $\Delta=1$ are independent of the spatial dimensionality.

IV. KINETIC GAUSSIAN MODEL ON HIERARCHICAL LATTICES WITH IRREDUCIBLE GENERATORS

Figure 3 shows three irreducible generators of hierarchical lattices. For the hierarchical lattice with a irreducible generator corresponding to Fig. 3(a), the fractal dimension $D_f = \ln 6 / \ln 2$ and the connectivity $Q=1$. Using assumption (23), the right-hand side of master equation (15) becomes

$$- \sum_i h_{q_i}(t) \left(\sigma_i - \frac{K}{b_{q_i}} \sum_{j(i)} \sigma_{j(i)} \right) P_e(\{\sigma\})$$
$$= - \sum_\alpha \left(1 - \frac{3K}{b_3} \right) \left[h_{2n_1}(t) \frac{\sigma_1^\alpha}{n_1} + h_{2n_2}(t) \frac{\sigma_2^\alpha}{n_2} + h_3(t)\sigma_3^\alpha + h_2(t)\sigma_4^\alpha + h_3(t)\sigma_5^\alpha \right] P_e(\{\sigma\})$$

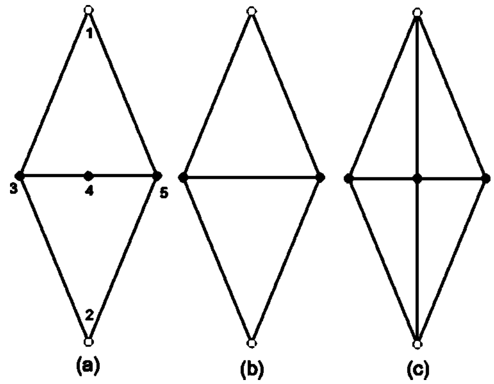

FIG. 3. Three irreducible generators of hierarchical lattices with different sets of parameters $\{A, C, b\}$ and $\{D_f, Q\}$: (a) $A=6$, $C=2$, $b=2$ and $D_f = \ln 6 / \ln 2$, $Q=1$; (b) $A=5$, $C=2$, $b=2$ and $D_f = \ln 5 / \ln 2$, $Q=1$; (c) $A=8$, $C=3$, $b=2$ and $D_f=3$, $Q=\ln 3/\ln 2$.

$$= - \sum_i \left(1 - \frac{3K}{b_3} \right) h_{q_i}(t) \sigma_i P_e(\{\sigma\}). \tag{45}$$

Thus, master equation (15) can be written in the following form:

$$\tau_0 \frac{d}{dt} \sum_i h_{q_i}(t) \sigma_i P_e(\{\sigma\}) = - \sum_i \left(1 - \frac{3K}{b_3} \right) h_{q_i}(t) \sigma_i P_e(\{\sigma\}). \tag{46}$$

In addition, the equilibrium probability distribution $P_e(\{\sigma\})$ can be expressed as

$$P_e(\{\sigma\}) = \frac{1}{Z} \exp\left(K \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i \frac{b_{q_i}}{2} \sigma_i^2 \right)$$
$$= \frac{1}{Z} \prod_\alpha \exp\left\{ K(\sigma_1^\alpha + \sigma_2^\alpha + \sigma_4^\alpha)(\sigma_3^\alpha + \sigma_5^\alpha) - \frac{b_{2n_1}}{2} \frac{(\sigma_1^\alpha)^2}{n_1} - \frac{b_{2n_2}}{2} \frac{(\sigma_2^\alpha)^2}{n_2} - \frac{b_2}{2} (\sigma_4^\alpha)^2 - \frac{b_3}{2} [(\sigma_3^\alpha)^2 + (\sigma_5^\alpha)^2] \right\}, \tag{47}$$

the renormalization-group transformation of which gives

$$\mathcal{R}\{P_e(\{\sigma\})\} = \frac{1}{Z} \prod_\alpha \int_{-\infty}^{+\infty} \prod_{i=3}^{5} d\sigma_i^\alpha \exp\left\{ K(\mu_1^\alpha + \mu_2^\alpha + \sigma_4^\alpha)(\sigma_3^\alpha + \sigma_5^\alpha) - \frac{b_{2n_1}}{2} \frac{(\mu_1^\alpha)^2}{n_1} - \frac{b_{2n_2}}{2} \frac{(\mu_2^\alpha)^2}{n_2} - \frac{b_2}{2} (\sigma_4^\alpha)^2 - \frac{b_3}{2} [(\sigma_3^\alpha)^2 + (\sigma_5^\alpha)^2] \right\}$$
$$= \frac{1}{Z} \prod_\alpha A_0 \exp\left[K' \mu_1'^\alpha \mu_2'^\alpha - \frac{b_{n_1}}{2} \frac{1}{n_1} (\mu_1'^\alpha)^2 - \frac{b_{n_2}}{2} \frac{1}{n_2} (\mu_2'^\alpha)^2 \right] = \frac{1}{Z'} \exp\left(K' \sum_{\langle ij \rangle} \mu_i' \mu_j' - \sum_i \frac{b_{q_i}}{2} \mu_i'^2 \right) = P_e'(\{\mu'\}), \tag{48}$$

where

$$A_0 = 2\pi\sqrt{\frac{2\pi}{b_3(b_2b_3 - 2K^2)}}, \tag{49}$$

the spins are rescaled as

$$\mu' = \xi(K)\mu = \sqrt{\frac{2b_2b_3 - 8K^2}{b_2b_3 - 2K^2}}\mu, \tag{50}$$

and the recursion relation of the renormalization-group transformation is given by

$$K' = R(K) = \frac{b_3K^2}{b_3^2 - 6K^2}, \tag{51}$$

from which one can get the critical point $K_c = b_3/3$. Using assumption (23), one can get

$$\mathcal{R}\{\sigma_1^\alpha P_e(\{\sigma\})\} = \frac{1}{\xi(K)}\mu_1'^{\alpha}P_e'(\{\mu'\}), \tag{52}$$

$$\mathcal{R}\{\sigma_2^\alpha P_e(\{\sigma\})\} = \frac{1}{\xi(K)}\mu_2'^{\alpha}P_e'(\{\mu'\}), \tag{53}$$

$$\mathcal{R}\{\sigma_3^\alpha P_e(\{\sigma\})\} = \mathcal{R}\{\sigma_5^\alpha P_e(\{\sigma\})\} = \frac{b_2K(\mu_1'^{\alpha} + \mu_2'^{\alpha})}{\xi(K)(b_2b_3 - 2K^2)}P_e'(\{\mu'\}), \tag{54}$$

and

$$\mathcal{R}\{\sigma_4^\alpha P_e(\{\sigma\})\} = \frac{2K^2(\mu_1'^{\alpha} + \mu_2'^{\alpha})}{\xi(K)(b_2b_3 - 2K^2)}P_e'(\{\mu'\}). \tag{55}$$

Thus, using assumption (23) and Eqs. (52)–(55), the renormalization-group transformation of the left-hand side of master equation (46) gives

$$\begin{aligned}
&\tau_0\frac{d}{dt}\mathcal{R}\left\{\sum_i h_{q_i}(t)\sigma_i P_e(\{\sigma\})\right\} \\
&= \tau_0\frac{d}{dt}\mathcal{R}\left\{\sum_\alpha\left[h_{2n_1}(t)\frac{\sigma_1^\alpha}{n_1} + h_{2n_2}(t)\frac{\sigma_2^\alpha}{n_2} + h_2(t)\sigma_4^\alpha + h_3(t)\right.\right. \\
&\quad\left.\left.\times(\sigma_3^\alpha + \sigma_5^\alpha)\right]P_e(\{\sigma\})\right\} \\
&= \tau_0\frac{d}{dt}\sum_\alpha\frac{2(b_2b_3 + 3b_2K - 4K^2)}{\xi(K)(b_2b_3 - 2K^2)} \\
&\quad\times\left(h_{n_1}(t)\frac{\mu_1'^{\alpha}}{n_1} + h_{n_2}(t)\frac{\mu_1'^{\alpha}}{n_2}\right)P_e'(\{\mu'\}) \\
&= \tau_0\frac{d}{dt}\sum_i\frac{2(b_2b_3 + 3b_2K - 4K^2)}{\xi(K)(b_2b_3 - 2K^2)}h_{q_i}(t)\mu_i'P_e'(\{\mu'\}) \\
&= \tau_0\frac{d}{dt}\sum_i h_{q_i}'(t)\mu_i'P_e'(\{\mu'\}),
\end{aligned} \tag{56}$$

where

$$h_{q_i}'(t) = \lambda h_{q_i}(t) = \frac{2(b_2b_3 + 3b_2K - 4K^2)}{\xi(K)(b_2b_3 - 2K^2)}h_{q_i}(t). \tag{57}$$

In the same way, the renormalization-group transformation of the right-hand side of master equation (46) gives

$$\begin{aligned}
&-\mathcal{R}\left\{\sum_i\left(1 - \frac{3K}{b_3}\right)h_{q_i}(t)\sigma_i P_e(\{\sigma\})\right\} \\
&= -\mathcal{R}\left\{\sum_\alpha\left(1 - \frac{3K}{b_3}\right)\left[h_{2n_1}(t)\frac{\sigma_1^\alpha}{n_1} + h_{2n_2}(t)\frac{\sigma_2^\alpha}{n_2}\right.\right. \\
&\quad\left.\left.+ h_2(t)\sigma_4^\alpha + h_3(t)(\sigma_3^\alpha + \sigma_5^\alpha)\right]P_e(\{\sigma\})\right\} \\
&= -\sum_i\frac{2(b_3 - 3K)(b_2b_3 + 3b_2K - 4K^2)}{\xi(K)b_3(b_2b_3 - 2K^2)} \\
&\quad\times h_{q_i}(t)\mu_i'P_e'(\{\mu'\}) \\
&= -\sum_i\left(1 - \frac{3K'}{b_3}\right)h_{q_i}''(t)\mu_i'P_e'(\{\mu'\}),
\end{aligned} \tag{58}$$

where

$$h_{q_i}''(t) = \omega h_{q_i}(t) = \frac{2(b_3 - 3K)(b_2b_3 + 3b_2K - 4K^2)}{\xi(K)(b_2b_3 - 2K^2)(b_3 - 3K')}h_{q_i}(t). \tag{59}$$

Furthermore, if we represent $h_{q_i}''(t)$ in terms of $h_{q_i}'(t)$ and perform the rescaling of the time scale by

$$\tau_0' = b^z\tau_0 = \frac{\lambda}{\omega}\tau_0 = \frac{b_3(b_3 + 3K)}{b_3^2 - 6K^2}\tau_0, \tag{60}$$

then the invariant form of master equation (46) can be restored—i.e.,

$$\tau_0'\frac{d}{dt}\sum_i h_{q_i}'(t)\mu_i'P_e'(\{\mu'\}) = -\sum_i\left(1 - \frac{3K'}{b_3}\right)h_{q_i}'(t)\mu_i'P_e'(\{\mu'\}). \tag{61}$$

From the recursion relation of the renormalization-group transformation—i.e., Eq. (51)—one can obtain the correlation-length critical exponent ν—i.e.,

$$\frac{1}{\nu} = \frac{1}{\ln b}\ \ln\left(\frac{dK'}{dK}\right)\bigg|_{K_c} = \frac{\ln 6}{\ln 2}. \tag{62}$$

Also, from Eq. (60) the dynamic critical exponent z can be calculated as

$$z = \frac{1}{\ln b}\ \ln\frac{b_3(b_3 + 3K)}{b_3^2 - 6K^2}\bigg|_{K_c} = \frac{\ln 6}{\ln 2}. \tag{63}$$

Thus, the above two equations (62) and (63) give

$$\Delta = z\nu = 1. \tag{64}$$

For hierarchical lattices with irreducible generators corresponding to Figs. 3(b) ($D_f = \ln 5/\ln 2$, $Q = 1$) and 3(c) ($D_f = 3$, $Q = \ln 3/\ln 2$), by means of the TDRG transformation, one can also, respectively, obtain that

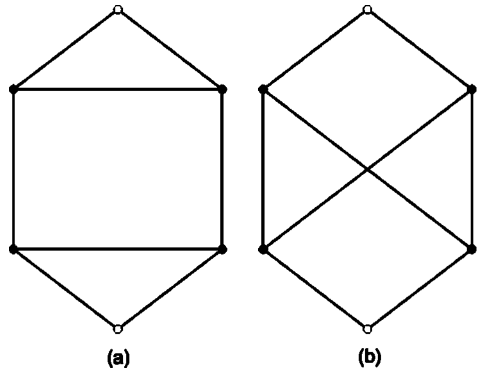

FIG. 4. Two irreducible generators of hierarchical lattices with the same set of parameters $\{A,C,b\}$ and $\{D_f,Q\}$: $A=8$, $C=2$, $b=3$ and $D_f=\ln 8/\ln 3$, $Q=\ln 2/\ln 3$.

$$z = \frac{\ln 5}{\ln 2}, \quad \Delta = 1, \tag{65}$$

and

$$z = \frac{\ln(16/3)}{\ln 2} = 4 - \frac{\ln 3}{\ln 2}, \quad \Delta = 1. \tag{66}$$

From the above calculations, it can be found that for the kinetic Gaussian model on hierarchical lattices with irreducible generators, which have different sets of parameters $\{A,C,b\}$ and $\{D_f,Q\}$, the dynamic critical exponent z has different values dependent on the concrete geometrical structure of these lattices, whereas the exponent Δ is the same—i.e., $\Delta=1$. In fact, this result is still correct even if hierarchical lattices with irreducible generators have the same set of parameters $\{A,C,b\}$ and $\{D_f,Q\}$. Figure 4 shows two irreducible generators of hierarchical lattices with the same A, C, b, and D_f, Q—i.e., $A=8$, $C=2$, $b=3$ and $D_f=\ln 8/\ln 3$, $Q=\ln 2/\ln 3$. For hierarchical lattices with irreducible generators corresponding to Figs. 3(a) and 3(b), the results of the TDRG transformation are, respectively,

$$z = 1 + \frac{\ln 4}{\ln 3}, \quad \Delta = 1, \tag{67}$$

and

$$z = \frac{\ln 10}{\ln 3}, \quad \Delta = 1. \tag{68}$$

V. CONCLUSION AND DISCUSSION

In this paper, by means of a real-space time-dependent renormalization-group transformation, we studied the critical slowing down of the kinetic Gaussian model on hierarchical lattices. The dynamic critical exponent z and the exponent Δ were calculated. For hierarchical lattices with reducible generators, both the dynamic critical exponent z and the exponent Δ are independent of the fractal dimension D_f of the lattice, the number of branches, m, and the number of bonds per branch, b, of the generator—i.e., $z=2$ and $\Delta=1$. For hierarchical lattices with irreducible generators, the exponent Δ is the same—i.e., $\Delta=1$; however, the dynamic critical exponent z is dependent on the concrete geometrical structure of these lattices and can be different even if the fractal dimension D_f and the connectivity Q are the same.

As mentioned in Sec. I, Zhu and Yang [2] exactly solved the kinetic Gaussian model on one-, two-, and three-dimensional translation-invariant lattices and found the exponent Δ independent of the spatial dimensionality—i.e., $\Delta=1$. By means of the TDRG transformation, Zhu and Yang [23] also studied the critical slowing down of the kinetic Gaussian model on the nonbranching, branching, and multibranching Koch curves and obtained that the exponent Δ is equal to 1 and is independent of the fractal dimension of the Koch curve. In this work, we also obtained that the exponent Δ is always 1 for the kinetic Gaussian model on hierarchical lattices with reducible or irreducible generators. Based on the above investigations, we may suppose that this result—i.e., $\Delta=1$—is produced by the kinetic Gaussian model itself and is irrelevant to the details of lattice structure. In other words, the result $\Delta=1$ seems to imply that the effect of lattice dependence of the dynamic critical exponent z counteracts that of the correlation-length critical exponent ν. We believe that there is some interesting physics behind it to be further exposed.

In this paper, we also found that the lattice dependence of the correlation-length critical exponent ν is the same as that of the dynamic critical exponent z. In the case of reducible generators, the correlation-length critical exponent ν is always equal to $1/2$, which is in good agreement with that in Ref. [37]. In the case of irreducible generators, this exponent is not equal to $1/2$, but has much dependence on the concrete geometrical structure of hierarchical lattices. As mentioned in Sec. I, for the Gaussian model on hierarchical lattices with reducible generators, the critical exponents γ and η, like ν, are also independent of the fractal dimension D_f of the lattice, while the others—i.e., α, β, and δ—are dependent on it [37]. Thus, it is obvious that the lattice dependence of the static critical exponents α, β, and δ is different from that of the dynamic critical exponent z. In fact, the behavior of the dynamic critical exponent z on hierarchical lattices reflects one aspect of universality for critical dynamics. As far as fractal lattices are concerned, universality is still an open problem even for static critical phenomena [36].

As we know, universality is one of the three pillars of modern critical phenomena [39], and it depends on a number of factors. The static critical phenomena depend on the spatial dimensionality and the symmetry of the order parameter, while the dynamic properties will depend on additional properties of the system which do not affect the statics. It is conjectured [4] that the universality class for critical dynamics is sufficiently determined by the conservation laws, Poisson-bracket relations among the order parameter, and the conserved densities, the spatial dimensionality, the symmetry of the order parameter, and any other properties that affect the static critical behavior. However, through the investigations of the dynamic critical exponent z in this paper, we have shown some unexpected difficulties in searching for the complete set of universality criteria for critical dynamics on hierarchical lattices. We believe this is an interesting question to be further studied. Also, the results of this paper suggest that as far as universality is concerned, one cannot learn much from hierarchical lattices to understand the behavior of critical dynamical exponents on regular lattices.

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